

## Some results on $K$ -algebras

Pramod K. Sharma

e-mail: pksharma1944@yahoo.com

School of Mathematics, Vigyan Bhawan, Khandwa Road,  
INDORE-452 017, INDIA.

### Abstract:

We give a new proof of the classical result due to Rodney Y. Sharp and Peter Vámos on the dimension of tensor product of a finite number of field extensions of a given field.

## 1 Introduction

Let  $K$  be a field. In this note, we prove some results on  $K$ -algebras. All rings and algebras are commutative with identity  $\neq 0$ . By the dimension of a ring  $A$  we mean the Krull dimension and denote it by  $\dim A$ . The transcendence degree of a field extension  $L/K$  shall be denoted by  $\text{trdeg}_K L$ . The results in this note grew while trying to understand the classical result on dimension of the tensor product of two field extensions proved in [6]. We first prove [Theorem 1] : Let  $R \subset A$  be rings where  $R$  is an integral domain with its field of fraction  $K$ . Then (1) If  $X_1, X_2, \dots, X_n$  are algebraically independent over  $A$  and  $A$  contains  $t_1, t_2, \dots, t_n$  algebraically independent over  $R$  then for  $L = K(X_1, \dots, X_n)$ ,  $\dim(L \otimes_R A) \geq n + \dim S^{-1}A$  where  $S$  is the multiplicatively closed subset  $R[t_1, \dots, t_n] - \{0\}$  of  $A$ , and (2) If  $X_1, X_2, \dots, X_n, \dots$  are algebraically independent over  $A$  and  $A$  contains  $t_1, t_2, \dots, t_n, \dots$  algebraically independent over  $R$  then for  $L = K(X_1, \dots, X_n, \dots)$ ,  $\dim(L \otimes_R A) = \infty$ . In Corollary 2.3, it is shown that equality holds in Theorem 1 under certain conditions. These results are used to find the dimension of the tensor product of a finite number of field extensions of a given field proved in [7]. Further, we give [Theorem 2.7] an alternative proof of the well known result that for an affine  $K$ -algebra  $A$  over a field  $K$ , for any non-zero-divisor  $f \in A$ ,  $\dim A = \dim A[1/f]$ .

## 2 Main Results

Before we prove that main results, let us recollect :

- (i) [5, Theorems 7.3 and 9.5]: If  $B$  is a faithfully flat  $A$ -algebra then  $\dim B \geq \dim A$ .
- (ii) [5, Exercise 9.2] If a ring  $B$  is an integral extension of a ring  $A$  then  $\dim A = \dim B$ .

We shall use these facts, whenever required, without further mention.

**Theorem 2.1.** *Let  $R \subset A$  be rings where  $R$  is an integral domain. Let  $K$  be the field of fractions of  $R$ . Then*

- (1) *If  $X_1, \dots, X_n$  are algebraically independent over  $A$  and  $A$  contains  $t_i$ ,  $i = 1, \dots, n$  algebraically independent over  $R$ , then*

$$\dim K(X_1, \dots, X_n) \otimes_R A \geq n + \dim S^{-1}A$$

where  $S = R[t_1, \dots, t_n] - \{0\}$ . Further, if  $A$  is Noetherian, then

$$\dim K(X_1, \dots, X_n) \otimes_R A \leq \dim A + n.$$

- (2) *If  $X_1, \dots, X_n, \dots$  are algebraically independent over  $A$  and  $A$  contains  $t_i$ ,  $i = 1, 2, \dots, n \dots$  algebraically independent over  $R$ , then*

$$\dim K(X_1, \dots, X_n, \dots) \otimes_R A = \infty.$$

*Proof.* (1) Let  $P'_0 \subsetneq P'_1 \subsetneq P'_2 \subsetneq \dots \subsetneq P'_m$  be a chain of prime ideals in  $S^{-1}A$ . Then there exist prime ideals  $P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \dots \subsetneq P_m$  in  $A$  such that  $P_i \cap S = \emptyset$  and  $S^{-1}P_i = P'_i$ . Note that

$$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \dots \subsetneq P_m \subsetneq (P_m, X_1 - t_1) \subsetneq \dots \subsetneq (P_m, X_1 - t_1, \dots, X_n - t_n)$$

is a chain of prime ideals in  $A[X_1, \dots, X_n]$ . If for  $T = R[X_1, \dots, X_n] - \{0\}$ ,  $T \cap (P_m, X_1 - t_1, \dots, X_n - t_n) \neq \emptyset$ , then there exist  $f(X_1, \dots, X_n) (\neq 0) \in R[X_1, \dots, X_n]$  such that

$$f(X_1, \dots, X_n) = g(X_1, \dots, X_n) + \sum (X_i - t_i)h_i(X_1, \dots, X_n)$$

where  $h_i \in A[X_1, \dots, X_n]$  and  $g(X_1, \dots, X_n) \in P_m[X_1, \dots, X_n]$ . This implies that  $f(t_1, \dots, t_n) = g(t_1, \dots, t_n) \in P_m$ . Since  $t_i$ 's are algebraically

independent over  $R$ ,  $f(t_1, \dots, t_n) \neq 0 \in P_m \cap S$ . This contradicts our assumption on  $P_i$ 's. Therefore  $T \cap (P_m, X_1 - t_1, \dots, X_n - t_n) = \phi$ , and

$$\dim T^{-1}(A[X_1, \dots, X_n]) \geq n + \dim S^{-1}A$$

where  $T = R[X_1, \dots, X_n] - \{0\}$ . Now, note that

$$R[X_1, \dots, X_n] \otimes_R A \cong A[X_1, \dots, X_n]$$

as  $R[X_1, \dots, X_n]$ -algebras. Hence

$$\begin{aligned} K(X_1, \dots, X_n) \otimes_{R[X_1, \dots, X_n]} A[X_1, \dots, X_n] &\cong T^{-1}A[X_1, \dots, X_n] \\ \Rightarrow \dim(K(X_1, \dots, X_n) \otimes_R A) &\geq n + \dim S^{-1}A. \end{aligned}$$

The final part of the statement is immediate since  $K(X_1, \dots, X_n) \otimes_R A$  is a localization of  $R[X_1, \dots, X_n] \otimes_R A$  which is isomorphic to  $A[X_1, \dots, X_n]$ . Further, as  $A$  is Noetherian,  $\dim A[X_1, \dots, X_n] = \dim A + n$  [5, Theorem 15.4]

(2) Let us note that

$$K(X_1, \dots, X_n, \dots) \otimes_{K(X_1, \dots, X_n)} (K(X_1, \dots, X_n) \otimes_R A) \cong K(X_1, \dots, X_n, \dots) \otimes_R A$$

Hence  $K(X_1, \dots, X_n, \dots) \otimes_R A$  is faithfully flat  $K(X_1, \dots, X_n) \otimes_R A$ -algebra.

Therefore

$$\begin{aligned} \dim K(X_1, \dots, X_n, \dots) \otimes_R A &\geq \dim K(X_1, \dots, X_n) \otimes_R A \\ &\geq n \quad (\text{use (1)}) \\ \Rightarrow \dim K(X_1, \dots, X_n, \dots) \otimes_R A &= \infty. \end{aligned}$$

□

**Remark 2.2.** In above Theorem, if  $B$  is any  $K(X_1, \dots, X_n)$ -algebra, then

$$\begin{aligned} \dim B \otimes_R A &\geq \dim K(X_1, \dots, X_n) \otimes_R A \\ &\geq n + \dim S^{-1}A \end{aligned}$$

Further, if  $B$  is  $K(X_1, \dots, X_n, \dots)$ -algebra, then

$$\dim B \otimes_R A = \infty.$$

These observations are immediate since  $B \otimes_R A$  is faithfully flat  $K(X_1, \dots, X_n) \otimes_R A(K(X_1, \dots, X_n, \dots) \otimes_R A)$ -algebra.

**Corollary 2.3.** *Let  $K$  be a field and  $A$  be a  $K$ -algebra. If  $X_1, \dots, X_n$  are algebraically independent over  $A$  and  $A$  contains a field extension of  $K$  of transcendental degree  $\geq n$ , then*

$$\dim K(X_1, \dots, X_n) \otimes_K A \geq n + \dim A.$$

*Further, if  $A$  is Noetherian, then*

$$\dim K(X_1, \dots, X_n) \otimes_K A = n + \dim A.$$

*Proof.* By assumption on  $A$ , there exist  $t_1, \dots, t_n$  algebraically independent over  $K$  such that  $K(t_1, \dots, t_n) \subset A$ . Hence for  $S = K[t_1, \dots, t_n] - 0$ ,  $S^{-1}A = A$ . Therefore, by the Theorem 1,

$$\dim K(X_1, \dots, X_n) \otimes_K A \geq n + \dim A.$$

Further, let  $A$  be Noetherian. Then as

$$K(X_1, \dots, X_n) \otimes_K A \cong T^{-1}A[X_1, \dots, X_n]$$

where  $T = [X_1, \dots, X_n] - 0$ , it is immediate that

$$\begin{aligned} \dim K(X_1, \dots, X_n) \otimes_K A &\leq \dim A[X_1, \dots, X_n] \\ &= n + \dim A \end{aligned}$$

Consequently

$$n + \dim A = \dim K(X_1, \dots, X_n) \otimes_K A.$$

□

**Theorem 2.4.** *Let  $L_i, i = 1, \dots, n$  be a field extension of a given field  $K$  and let  $\text{trgdeg}_K L_i = t_i$ . Assume  $t_1 \leq t_2 \leq \dots \leq t_{n-1} \leq t_n$ . If  $t_{n-1} < \infty$  then*

$$\dim(L_1 \otimes_K \dots \otimes_K L_n) = t_1 + t_2 + \dots + t_{n-1},$$

*otherwise*

$$\dim(L_1 \otimes_K \dots \otimes_K L_n) = \infty.$$

*Proof.* We shall consider the two cases separately.

Case 1.  $t_1 \leq t_2 \leq \dots \leq t_{n-1} < \infty$ .

Let  $B_k = \{x_{k1}, x_{k2}, \dots, x_{kt_k}\}$  be a transcendental basis of  $L_k$  over  $K$  for  $k = 1, 2, \dots, n-1$ . Put  $E_k = K(x_{k1}, x_{k2}, \dots, x_{kt_k})$ . Then  $E_k/K$  is purely

transcendental field extension of transcendental degree  $t_k$  and  $L_k/E_k$  is algebraic. Hence

$$E_1 \otimes_K E_2 \otimes_K \cdots \otimes_K E_{n-1} \otimes_K L_n \xrightarrow{i_1 \otimes \cdots \otimes i_{n-1} \otimes Id} L_1 \otimes_K L_2 \otimes_K \cdots \otimes_K L_n,$$

where  $i_k : E_k \hookrightarrow L_k$  is inclusion map for  $k = 1, \dots, n-1$  and  $Id$  is identity map, is an integral extension. Therefore

$$\dim(L_1 \otimes_K \cdots \otimes_K L_n) = \dim(E_1 \otimes_K E_2 \otimes_K \cdots \otimes_K E_{n-1} \otimes_K L_n).$$

Let  $Y_{11}, Y_{12}, \dots, Y_{1t_1}, Y_{21}, \dots, Y_{2t_2}, \dots, Y_{(n-1)1}, \dots, Y_{(n-1)t_{(n-1)}}$  be algebraically independent elements over  $K$ . Then for  $F_k = K(Y_{11}, \dots, Y_{1t_k}), k = 1, \dots, n-1$ , we have

$$F_1 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n \cong E_1 \otimes_K \cdots \otimes_K E_{n-1} \otimes_K L_n$$

Therefore

$$\dim(F_1 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n) = \dim(L_1 \otimes_K \cdots \otimes_K L_n)$$

Let us note that  $F_2 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n$  is a localization of

$$L_n[Y_{21}, \dots, Y_{2t_2}, \dots, Y_{n-1,1}, \dots, Y_{(n-1)t_{(n-1)}}]$$

over a multiplicatively closed subset, hence is a Noetherian ring. Therefore by Corollary 2.3,

$$\dim(F_1 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n) = t_1 + \dim(F_2 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n).$$

By successive application of the Corollary 2.3 or by induction it is immediate that

$$\dim F_2 \otimes_K \cdots \otimes_K F_{n-1} \otimes_K L_n = t_2 + \cdots + t_{n-1}$$

Hence in this case the result follows.

Case 2.  $t_{n-1} = t_n = \infty$ .

First of all, note that for any  $\sigma \in S_n$

$$L_1 \otimes_K \cdots \otimes_K L_n \cong L_{\sigma(1)} \otimes_K \cdots \otimes_K L_{\sigma(n)}.$$

Therefore

$$L_1 \otimes_K \cdots \otimes_K L_n \cong L_n \otimes_K L_{n-1} \otimes_K \cdots \otimes_K L_2 \otimes_K L_1.$$

Put  $B = L_{n-1} \otimes_K \cdots \otimes_K L_2 \otimes_K L_1$ . Then

$$\dim(L_1 \otimes_K \cdots \otimes_K L_n) = \dim L_n \otimes_K B.$$

By assumption  $B$  contains infinite algebraically independent elements over  $K$ . Hence the result is immediate from Theorem 1(2). □

**Remark 2.5.** If  $A_i, i = 1, \dots, n$  denote integral extension of  $L_i$ , then

$$\dim A_1 \otimes_K \cdots \otimes_K A_n = \dim L_1 \otimes_K \cdots \otimes_K L_n.$$

Further, if  $A_i$  is any  $L_i$ -algebra, then

$$\dim A_1 \otimes_K \cdots \otimes_K A_n \geq \dim(L_1 \otimes_K \cdots \otimes_K L_n).$$

**Lemma 2.6.** Let  $K[X_1, \dots, X_n] = K[\underline{X}]$  be a polynomial ring in  $n$ -variables  $X_i, i = 1, \dots, n$  over a field  $K$ . Then for any  $f(\neq 0) \in K[\underline{X}]$ ,  $\dim K[\underline{X}, 1/f] = n$ .

*Proof.* Let  $\overline{K}$  be the algebraic closure of  $K$ . Then, since  $\overline{K}[\underline{X}, 1/f]$  is integral over  $K[\underline{X}, 1/f]$ , we have

$$\dim \overline{K}[\underline{X}, 1/f] = \dim K[\underline{X}, 1/f].$$

Hence, to prove the result, we can assume that  $K$  is algebraically closed. Note that  $\dim K[\underline{X}] = n$  and for the multiplicatively closed subset  $S = \{f^t | t \geq 0\}$ ,  $S^{-1}K[\underline{X}] = K[\underline{X}, 1/f]$ . Since  $f \neq 0$ ,  $f$  does not vanish on  $K^n$ . Thus, if for  $\underline{\lambda} = \lambda_1, \dots, \lambda_n$  in  $K^n$ ,  $f(\underline{\lambda}) \neq 0$ , then for the maximal ideal  $M = (X_1 - \lambda_1, \dots, X_n - \lambda_n)$  in  $K[\underline{X}]$ ,  $M \cap S = \emptyset$ . Therefore  $S^{-1}M$  is a maximal ideal in  $S^{-1}K[\underline{X}]$ . Clearly, height of  $M$ , i.e.  $ht M = n = ht S^{-1}M$ . Therefore  $\dim K[\underline{X}, 1/f] = n$ . □

**Theorem 2.7.** Let  $A$  be an affine algebra over a field  $K$ . Then for any non-zero-divisor  $f$  in  $A$ ,  $\dim A[1/f] = \dim A$ .

*Proof.* Let  $A = \frac{K[X_1, \dots, X_n]}{I}$ . Since  $f$  is a non-zero-divisor in  $A$ ,  $f$  lies in no prime ideal associated to  $I$  in  $K[X_1, \dots, X_n]$ . Let  $p$  be an associated prime ideal of  $I$  in  $K[X_1, \dots, X_n]$  such that

$$\dim A = \dim \frac{K[X_1, \dots, X_n]}{p}.$$

Then  $\bar{f}$ , image of  $f$  in  $\frac{K[X_1, \dots, X_n]}{p}$ , is non-zero. Note that  $\dim A[1/f] \leq \dim A$ . Further, as  $\frac{K[X_1, \dots, X_n]}{p} \cdot [1/\bar{f}]$  is a quotient ring of  $A[1/f]$  in a natural way,

$$\dim A[1/f] \geq \dim \frac{K[X_1, \dots, X_n]}{p} [1/\bar{f}].$$

Thus to prove Theorem, it is sufficient to show that

$$\dim \frac{K[X_1, \dots, X_n]}{p} = \dim \frac{K[X_1, \dots, X_n]}{p} [1/\bar{f}].$$

Let us observe that

$$\begin{aligned} \theta : \frac{K[X_1, \dots, X_n][Y]}{(p, fY - 1)} &\rightarrow \frac{K[X_1, \dots, X_n]}{p} [1/\bar{f}] \\ Y &\mapsto 1/\bar{f} \end{aligned}$$

is  $\frac{K[X_1, \dots, X_n]}{p}$  algebra isomorphism. Therefore

$$\dim \frac{K[X_1, \dots, X_n]}{p} [1/\bar{f}] = \dim \frac{K[X_1, \dots, X_n][Y]}{(p, fY - 1)}.$$

We note that  $fY - 1 \notin p[Y]$ . As  $\frac{K[X_1, \dots, X_n]}{p} [1/\bar{f}]$  is an integral domain, the ideal  $(p, fY - 1)$  is prime in  $K[X_1, \dots, X_n, Y]$ . Now, note that  $K[X_1, \dots, X_n, Y]$  is a Cohen-Macaulay ring of dimension  $n + 1$ . By [4, Ex. 19, page 104],  $ht(p, fY - 1) = htp + 1$ . Therefore

$$\begin{aligned} \dim \frac{K[X_1, \dots, X_n][Y]}{(p, fY - 1)} &= (n + 1) - (htp + 1) \\ &= n - htp \\ &= \dim \frac{K[X_1, \dots, X_n]}{p}. \end{aligned}$$

Thus  $\dim A = \dim A[1/f]$ . □

We, now, deduce the following well known result:

**Corollary 2.8.** *Let  $A$  be an affine algebra over a field  $K$  which is an integral domain. Then  $\dim A = \text{trdeg}_K L$  where  $L$  is the field of fractions of  $A$ .*

*Proof.* Let  $\{y_1, \dots, y_s\}$  be a maximal algebraically independent set of elements in  $A$  over  $K$ . Then every  $a \in A$  is algebraic over  $K[y_1, \dots, y_s]$ . Since  $A$  is an affine algebra over  $K$ ,  $A = K[a_1, \dots, a_t]$  for some  $a_i, i = 1, 2, \dots, t$ . Since each  $a_i$  is algebraic over  $K[y_1, \dots, y_s]$  there exists an element  $f (\neq 0) \in K[y_1, \dots, y_s]$  such that  $A[1/f]$  is integral over  $K[y_1, \dots, y_s][1/f]$ . Thus

$$\begin{aligned} \dim A[1/f] &= \dim K[y_1, \dots, y_s][1/f] \\ &= s \quad (\text{Lemma 2.6}) \end{aligned}$$

Therefore by Theorem, it is immediate that  $\dim A = \text{trdeg}_K L$ .  $\square$

### Acknowledgement

The author is very thankful to R.Y. Sharp for sending reprints of his articles.

### REFERENCES

1. M.F. Atiyah, I.G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley Publ. Co., 1969.
2. David Eisenbud, Commutative Algebra with a view Toward Algebraic Geometry, Springer-Verlag, New York, Inc., 1995.
3. Arno van den Essen, Polynomial automorphisms and the Jacobian conjecture, Progress in Mathematics, Vol 190, Birkhäuser, 2000.
4. Irving Kaplansky, Commutative Rings, The University of Chicago Press, Chicago, 1974.
5. Hideyuki Matsumura, Commutative Ring Theory, Cambridge University Press, 1986.
6. Rodney Y. Sharp, Dimension of the tensor product of two field extensions, Bulletin London Math. Soc. 9(1977), 42-48.
7. Rodney Y. Sharp and Peter Vamos, The dimension of the tensor product of a finite number of field extensions., Jour. of Pure and Applied Algebra, 10(1977), 249-252.